On the quantum Hall effect

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Abstract. *We consider a particle in a 2 dimensional plane in a periodic potential and a homogeneous magnetic field perpendicular to the plane. Kubo's expression for conductivity of the Hall current is an integer.*
This result of Thouless, Kohomoto, Nightingale and den Nijs is interpreted geometrically.

1. THE EXPERIMENT OF V. KLITZING, DORDA AND PAPPER

In 1980 v. Klitzing, Dorda and Pepper published an article entitled «New Methods for High Accuracy Determination of the Fine Structure Constant Based on the Quantized Hall Resistance» [1]. They measured the conductivity σ_H of the Hall-current in the twodimensional boundary layer of a Silicon-Siliconoxide transistor with the magnetic field perpendicular to the layer. The magnetic field had the order of magnitude 100 Kilogauss and the temperature was about 1.5°K. In latter experiments [e.g.2], σ_H was measured as function of the magnetic field. The qualitative behavior of conductivity is represented by the graph in fig. 1.

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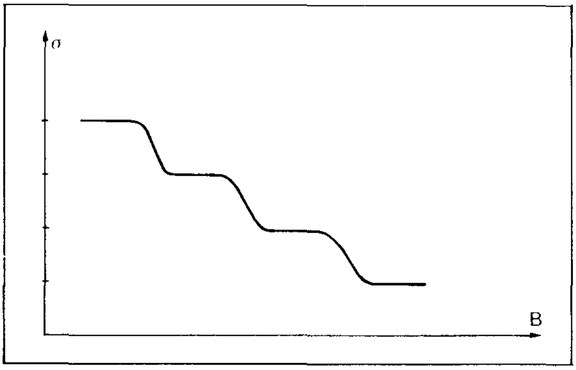

Fig. 1.

It shows the characteristic flat pieces, the so called plateaus where conductivity takes the values $\sigma_H = n\,\sigma$, n is an integer and $\sigma := e^2/h$ (in atomic units $m_e = e = \hbar = 1$, $\sigma = 1/2\pi$). The formula represents experimental data with the precision $1 : 10^8$.

In some later experiments it was found that there are smaler plateaus at fractional values of σ.

The same qualitative result has been found in other systems.

The problem with the interpretation of this experimental is twofold: Firstly, how do these integer valued conductivities come about and secondly, why is the result not sensitive to dirt or defect in the lattice.

In the following we present a geometrical reformulation of the work of Thouless, Kohmoto, Nightingale and den Nijs [2]. Related and additional material can be found in [3].

2. ONE ELECTRON IN A HOMOGENEOUS MAGNETIC AND ELECTRIC FIELD

In this chapter we shall discuss the Schrödinger operator for one electron moving in a plane with a homogeneous magnetic field perpendicular to the plane and a homogeneous electric field in the plane. (We choose B and E in the direction of the 3 and 1-axis respectively). It turns out that the use of gauge invariant objects to formulate this problem leads to a very simple analysis and allows the integration of Heisenbergs equation of motion for the position and velocity operators in a straight forward manner [3]. In the following we shall identify vectors $x \in \mathbb{R}^2$ and $(x, 0) \in \mathbb{R}^3$.

The Schrödinger operator is given (atomic units) by

$H := \frac{1}{2} v^2 - Ex$, E electric field, x position in the 1 - 2 plane

$v = p - eA$ velocity operator in the 1 - 2 plane.

A electromagnetic field strength ($B = \operatorname{rot} A$).

In a particular gauge $A = \frac{1}{2} B \wedge x$.

The operator $H_0 := \frac{1}{2} v^2$ has the integral of motion $c := x + v \wedge b$, $b := = B/|B|^2$. c is the operator of the Landau center (We set the 3-component to zero). Its interpretation is made simple by the following formula

$$x(t) := e^{iH_0 t} x e^{-iH_0 t} = c + R(t) r.$$

$$R(t) := \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}, \omega = |B|,$$

$$r := v_0 \wedge b.$$

v_0 is the time zero velocity of the particle. The equation tells us the well known fact that an electron in a homogeneous field rotates around the Landau center with the Larmor frequency $\omega = |B|$. $x(t)$ is the integral of the Heisenberg equation of motion and is the result of a straightforward and simple computation starting from the commutation relations

$$[\alpha v, \beta v] = -i(\alpha \wedge \beta \cdot B), (\alpha, \beta \in \mathbb{R}^2),$$

$$[\alpha c, \beta c] = -i(\alpha \wedge \beta \cdot b),$$

$$[\alpha v, \beta c] = 0.$$

Notice that the two nontrivial components of v do not commute; in fact they obey Heisenbergs commutation relations. Hence H_0 is the Schrödinger operator of a harmonic oscillator say in the variable x $\left(v_1 \text{ multiplication operator } x, v_2 = = iB \frac{\mathrm{d}}{\mathrm{d}x}\right)$.

Even in the presence of a homogeneous electric field the Heisenberg equation of motion for c and v can still be integrated [3]; they read

$$\begin{aligned} \alpha \dot{c} &= i[H, \alpha c] = i[H_0, \alpha c] - i[Ex, \alpha c] \; (\alpha \in \mathbb{R}^2) \\ &= -i[Ex, \alpha c] \\ &= -i[E(c - v \wedge b), \alpha c] \\ &= -E \wedge \alpha \cdot b = \alpha \cdot D, D := E \wedge b \end{aligned}$$

and

$$\begin{aligned}\dot{v} &= i\,[H_0, v] - i\,[Ex, v] \\ &= B \wedge v - E.\end{aligned}$$

Both equations can easely be integrated,

$$\begin{aligned}c\,(t) &= Dt + c_0 \\ v\,(t) &= R(t)\, v_0 + D.\end{aligned}$$

Inserting these expressions into the definition of the operator of the Landau center c yields the following formula for the position operator:

$$x\,(t) = (c_0 - D \wedge b) - R\left(t - \frac{\pi}{2\omega}\right) v_0 + Dt.$$

The equation has the following interpretation: The electron circles around its mean position which moves with uniform speed D perpendicular to E and B. This situation differs qualitatively with the one without magnetic field, where the velocity of the electron grows linearly in time under the influence of the electric field.

To end this section we give the following heuristic argument for the Hall conductivity: The current of non interacting electrons in space is given by the expression $j =$ density. (mean velocity of one electron) and related to E by Ohm's law $j = \sigma_H E$. The mean velocity is by the previous expression for $v(t)$ just the drift D. To compute the density of particles, we resort to the following argument: c_x and c_y play a role analogous to position and momentum. Hence the unit in phase space for one electron is not $h = 2\pi$ but $\frac{2\pi}{B}$ due to the CCR. The density turns out to be the reciprocal if all electrons are put into the groundstate of $H_0 = \frac{1}{2} v^2$ (harmonic oscillator) considered on functions of v_x only; otherwise we get for the density $n \cdot \frac{B}{2\pi}$, where n denotes the number of harmonic oscillator wave functions (Landaulevels) occupied. Putting this together one gets the desired result $\sigma_H = n/2\pi$.

3. ONE ELECTRON IN A HOMOGENEOUS MAGNETIC FIELD AND A PERIODIC POTENTIAL

In this chapter we present the Bloch analysis for a Schrödinger operator with a homogeneous magnetic field and a periodic potential. The results-although not new [4, 5] - have not yet found their way into textbooks.

1. The generalized momentum: The main point in the analysis is the existence of a generalized momentum operator $\tilde{p}$ which is defined by $\tilde{p} := B \wedge c$. From the commutation relations of c one gets

$$[\alpha\tilde{p}, \beta\tilde{p}] = i\alpha \wedge \beta \cdot B \quad (\alpha, \beta \in \mathbb{R}^3, \alpha_3 = \beta_3 = 0)$$

$$[\alpha\tilde{p}, \beta v] = 0.$$

Consider now the unitary group generated by $\tilde{p}$,

$$T(a) := \exp - ia\tilde{p} \quad (a \in \mathbb{R}^2),$$

It statisfies the Weyl commutation relations

$$T(u)\, T(v) = T(u + v) \cdot \exp \frac{i}{2} (u \wedge v \cdot B) \quad (u, v \in \mathbb{R}^2)$$

$T(u)$ and $T(v)$ commute if the flux of the magnetic field through the surface spanned by u, v ($u_3 = v_3 = 0$) is 2π times an integer.

Sometimes it is usefull to have an explizit version of $T(u)$. In the gauge previously mentioned, where

$$A = \frac{1}{2} B \wedge x, \text{ one gets}$$

$$T(u) = e^{-iup} \cdot e^{-iuA}$$
$$= e^{-iuA} e^{-iup}.$$

To get this results we used the statements: If $[A, B]$ commutes with A and B then $e^{A+B} = e^A e^B e^{1/2[A,B]}$ and $[up, uA] = o$ ($u \in \mathbb{R}^2$). For a general gauge $T(u)$ has always the structure translation operator times multiplicative phase. This fact will be crucial in the following Bloch wave analysis.

2. Translation symmetries of the Schrödinger operator: Consider the Schrödinger operator for one electron in a homogeneous magnetic field and a periodic potential V,

$$V(x + a) = V(x) \quad (a \in L \subset \mathbb{R}^2, x \in \mathbb{R}^2).$$

L is a lattice generated by l_1 and l_2.

V shall be a continuous realvalued bounded function,

$$H := \frac{1}{2} v^2 + V, \; D(H) = C_0^\infty(\mathbb{R}^2).$$

H is essentially selfadjoint [6]. (Of course the conditions on V are not optimal) Due to the general structure of $T(u)$ just mentioned above, $T(u)$ commutes with H if $u \in L$,

$$[H, T(u)] = 0 \quad (u \in L).$$

Since the flux through a unit cell of L is generally not an integer multiple of 2π two generalized translations $T(u)$ and $T(v)$ $(u, v \in L)$ will not commute. However the following statement holds:

LEMMA: *Let H and $T(u)$ be the operators previously defined and let M be a sublattice of L generated by the vectors m_1, m_2. Then the following statement holds: If the flux $\phi := l_1 \wedge l_2 \cdot B$ is a rational number times 2π then the set of operators $\{H, T(u) \mid u \in M\}$ is abelian for M properly chosen.*

We shall call M the magnetic lattice.

Proof: By assumption $\phi = 2\pi \dfrac{p}{q}$, $p, q \in Z$. Choose for example $m_1 = l_1, m_2 = q l_2$. Now the statement is clearly correct.

3. The Bloch analysis of the Hilbert space of states: In this section we describe a unitary mapping U of the space of states $H = L^2(\mathbb{R}^2, \mathrm{d}x)$ onto a direct integral of Hilbert spaces

$$\tilde{H} := \int \frac{\mathrm{d}^2 k}{|C^*|} \; L_k^2(C, \mathrm{d}x),$$

where $L_k^2(C, \mathrm{d}x)$ $(k \in C^*)$ is the space of square integrable functions over the unit cell $C := \mathbb{R}^2/M$ and the integral ranges over $C^* := \mathbb{R}^2/M^*$. (M^* denotes the dual lattice of M). The result is summarized in the following

LEMMA: *The mapping U defined by*

$$U : f \to f_k := \sum_{m \in M} e^{-ikm} \, T(m) f \quad (f \in S(\mathbb{R}^2), k \in \mathbb{R}^2)$$

extends by continuity to a unitary map of H onto $\tilde{H}$.

The proof is quite simple and we shall give two main pieces of it: i) Isometry of U follows from the following argument for which we use the gauge $A = \frac{1}{2} B \wedge x$:

$$\begin{aligned}\langle Uf, Uf\rangle &= \int_{C^*} \frac{\mathrm{d}^2 k}{|C^*|} \sum_{m,m'} e^{ik(m-m')} \int_C \mathrm{d}^2 x\, e^{iA(m-m')}\, \overline{f(x+m)}\, f(x+m') \\ &= \sum_{m,m'} \int_C \mathrm{d}^2 x\, e^{iA(m-m')}\, \overline{f(x+m)}\, f(x+m') \int_{C^*} \frac{\mathrm{d}^2 k}{|C^*|} e^{ik(m-m')} \\ &= \sum_m \int_C \mathrm{d}^2 x\, |f(x+m)|^2 \\ &= \langle f, f\rangle.\end{aligned}$$

Since f is a Schwartz test function the order of integrations can be changed.

ii) The adjoint and inverse of U is given by the expression

$$U^* : f_k \longrightarrow f(x) := \int_{C^*} \frac{\mathrm{d}^2 k}{|C^*|} e^{ikm} (T(-m) f_k)(x) \qquad (x - m \in C, m \in M).$$

In order to be well defined we have to identify $f_k(x)$ $(x \in C, k \in C^*)$ with its periodic extension in the variable $x \in \mathbb{R}^2$ and to assume regularity in k and x. To check that U^* defined by the equation above is the adjoint of U it is usefull to note that $A(x)m = A(x+m)m$ $(m \in \mathbb{R}^2)$. Notice that $f_k(x) = Uf(x)$ $(f \in S(\mathbb{R}^2))$ has the following periodicity properties:

i) $f_{k+m^*}(x) = f_k(x)$ $(x, k \in \mathbb{R}^2, m^* \in M^*)$

ii) $(T(m) f_k)(x) = e^{ikm} f_k(x)$ $(x, k \in \mathbb{R}^2, m \in M)$.

In the special gauge mentioned previously the second periodicity property takes the form

ii′) $f_k(x+m) = e^{ikm} e^{imA(x)} f_k(x)$.

ii) shows that f_k is a generalized Bloch function.

4. The Bloch analysis of the Schrödinger operator: The direct integral decomposi-

tion of the last section is well adapted to the analysis of H; to explain that, we introduce the operator valued function $H(k)$ acting on a domain $D_k \subset L_k^2$ defined as follows. Consider the functions

$$\tilde{D}_k := \{f \mid f \in H^2_{\text{loc}}(\mathbb{R}^2),\ T(m)f = e^{ikm}f,\ m \in M\}.$$

and their restrictions to C

$$D_k := \{g \mid g = f|_C,\ f \in \tilde{D}_k\}.$$

Manifestly the functions satisfy the boundary condition ii) mentioned in the last section. The operator $H(k)$ has by definition the same symbol as H and the domain D_k:

$$H(k) := \frac{1}{2} v^2 + V,\ D(H) = D_k \subset L_k\ (k \in \mathbb{R}^2).$$

It has the properties:

THEOREM: $H(k)$ *(defined above) is selfadjoint, realanalytic and periodic in* k

$$H(k+m) = H(k) \quad (k \in \mathbb{R}^2, m \in M).$$

Furthermore it is related to $\overline{H}$ *by*

$$(*) \qquad U\overline{H}U^{-1} = \int_C \frac{d^2k}{|C^*|} H(k),$$

where U *denotes the unitary mapping introduced in the previous sections.*

Proof: Since V is a bounded perturbation of the selfadjoint operator $\frac{1}{2}v^2$, $H(k)$ ist selfadjoint with the same domain. Analyticity is easyly seen by the following argument: Consider the unitary operator valued realanalytic function $B(k) := = e^{ikx}$. Then $B(k)\ H(k)\ B(k)^{-1} = \frac{1}{2}(v+k)^2 + V(x)$ with domain $D_{k=0}$ is manifestly analytic in k. Periodicity follows from periodicity of the domain of definition D_k. To prove the last part of the statement it is enough to show equality (*) on $S(\mathbb{R}^2)$ since this is a core for H. Let f be a Schwartz test function, then

$$(Hf)(x) = H \int \frac{d^2k}{|C^*|} e^{ikm}(T(-m)f_k)(x), \quad \begin{array}{l} x - m \in C \\ m \in M \end{array}$$

$$= \int \frac{d^2 k}{|C^*|} e^{ikm} (T(-m)\, H(k) f_k)(x).$$

4. GEOMETRICAL INTERPRETATION OF THE HALL CONDUCTIVITY

1. Linear response theory yields an expression for the Hall conductivity which is Kubo's formula

$$\sigma_H = \frac{i}{2\pi} \sum_{i=1}^{n} \int_{C^*} \left(\frac{df^i}{dk_1} k,\ \frac{df^i}{dk_2} k \right) dk_1 \cdot dk_2$$

n denotes the «number of bands occupied», f_k^i is the i-th eigenfunction of $H(k)$, and C^* is the unit cell of M^*.

There is only a formal argument which supports the validity of this formula. It would be very nice to have a proof. However this is difficult. There is also a mathematical problem with the integrand in the above expression for σ_H. Implicitly it is assumed, that there exist sufficiently regular eigenfunctions f_k^i, $k \in C^*$. However this is in general not correct, not even locally [7]. Therefore it is very usefull and nice to have an expression for the conductivity which only involves the projector $P(k)$ onto the eigenstates f_k^i since $P(k)$ is real analytic ($k \in \mathbb{R}^2$) provided the n-th eigenvalue is well separated from the $n + 1$ st one. This will be always assumed in the following:

Assumption: If $k \in C^*$ then $E_{n+1}(k) > E_n(k)$, where $E_n(k)$ denotes the n-the eigenvalue of $H(k)$, counted from below.

LEMMA: *Let $H(k)$ be the Schrödinger operator as defined in the preceeding chapter. Let the n first eigenvalues of $H(k)$ be non degenerate for k in a neighborhood N of k_0. Let f_k^j be an analytic choice of eigenvectors with total projector $P(k)$. Then the following identity holds*

$$\sum_{i=1}^{n} (df^i, df^i) = \text{Trace } dP\, P\, dP.$$

We shall not give a proof of the statement because it involves straightforward computation.

2. The Hilbert space $\tilde{H}$ introduced in the previos chapter has the structure of a

Hilbert bundle over the torus T (T is the same as C^* up to the boundary; for geometrical concepts we refer to appendix C of ref. [8]). On $\tilde{H}$ there is a natural unitary parallel transport mapping D_k onto $D_{k'}$,

$$P(k, k') : g_k \longrightarrow h_k(x) := \exp i (k - k')\, x \cdot g_k(x).$$

The corresponding connection turns out to be $D = \nabla_k - ix$. It provides the tool to construct an induced connection on vector bundles over T.

3. Consider the projector valued function $P(k)$, $k \in T$, onto the first n eigenfunctions of $H(k)$. Let B^n be the $\mathbb{C}^n$ - vector bundle over T, where the fiber on top of $k \in T$ is just the range of $P(k)$. On B^n there is the following natural connection, defined on local coordinates (sections) $\{f_i\}_{i=1}^n$ by

$$\nabla (f^i) := \sum_j \omega^{ij} f^j.$$

$$\omega^{ij} := \langle f^i D f^j \rangle.$$

The curvature is given by the general formula $\Omega = \mathrm{d}\omega - \omega \wedge \omega$. Since the basis T is 2-dimensional only the first term contributes, hence

$$\Omega = \mathrm{d}\omega, \ \Omega(f^i) = \sum_j \Omega^{ij} f^j, \ \Omega^{ij} = \mathrm{d}\omega^{ij}.$$

4. On the torus there are canonically constructed closed forms, the Chern classes. The first Chern class is defined by

$$C_1 := \frac{1}{2\pi i} \text{Trace}\ \Omega \left(= \sum_{i=1}^{n} \Omega^{ii} \right).$$

The integral of C_1 over the torus is an integer.

Now we want to show that C_1 is equals the integrand in the Kubo formula if the f_k^i are the first n eigenfunctions of $H(k)$. By definition Ω^{ii} is locally given by

$$\Omega^{ii} = \mathrm{d} \langle f^i D f^i \rangle .$$

$$= \langle \mathrm{d} f^i , \mathrm{d} f^i \rangle .$$

To get the last identity we used the invariance of Ω^{ij} under the substitution $f_k(x) \longrightarrow e^{ikx} f_k(x)$ and the identity $D = e^{ikx} \mathrm{d}\, e^{-ikx}$. Hence Trace Ω equals σ_H as given at the beginning of this section.

Of course there is also a much more elementary derivation of the result about quantization of conductivity. This is the original argument of Thouless, Kohmoto, Nightingale and den Nijs [2]. It runs roughly as follows: Let f_k^i $(i = 1, \ldots, n)$ be the first n eigenfunctions of $H(k)$ defined on $\mathbb{R}^2$ (not on T). f_k and $f_{k'}$ differ by a phase if k and k' are two elements of δC^* identified as elements of the torus. Summing up the phases along the boundary of C^* gives necessarily $2\pi i$ times an integer. This is the conductivity. The transition from the surface integral in the Kubo formula to the line integral over C^* is made possible by the fact that the first Chern class-considerd as a two form an $\mathbb{R}^2$-is not only closed but exact. Hence Stokes theorem can be applied.

The topological quantum numbers found by Thouless, Kohmoto, Nightingale and den Nijs turn out to be the only quantized quantities associated with the energy bands [10]. It is remarkable that they turn out to be the measurable conductivities.

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